Two-phase Method and Theorems on Minimization of Distance

(A presentation on Mathematical Programming)

Presented By:
Kshitiz Mangal Bajracharya (9)
Anjana Bhujel (63)
Dilli Raj Bhatta (33)
Amar Singh Chaudhary (47)
Prem Chaudhary (21)
Bhupal Chamrekar (77)

Two-Phase Method - Why do we need it?

If the **all constraints of the given LPP has $\leq$ sign** for the comparison, then introduction of slack variables will not create any hurdles in order to choose a basis for obtaining the initial basic feasible solution (BFS). So, **we may easily choose a basis for obtaining initial BFS in this case**.

But, if **it involves other signs for comparison**, say for example $\geq$ constraints, the introduction of surplus variables might create some hindrances to choose the initial BFS. **Two-phase method helps us to choose a basis to obtain the initial BFS in such cases**.

How to use Two-Phase Method?

- **Phase 1**
  - Change the given LPP into **standard form** so that simplex method can be used.
  - Now, to each of the constraints, add a variable. That is to say if a constraint is, for example, $2x + 3y = 5$, make it $2x + 3y + A = 5$. Such a variable A is called an **artificial variable**, which is necessarily non-negative.
  - Introduce the objective function $w = A_1 + A_2 + \cdots + A_p$, where each of the A's are artificial variables.
  - We now try to **minimize** w subjected to the **new constraints we obtained by adding the artificial variables**, using the usual simplex method. Apply row operations to the row that represents original objective function as well, for easier computations.

- The following cases may arise.
    - **The objective function w has been minimized with a positive value and the artificial variables are in the basis at a positive level (that is, with a non-zero value)**. This means the **given LPP has no solution**.
    - **The objective function w has been minimized with $w_{\min} = 0$ and the artificial variables are not in the basis**. This means **an initial BFS has been obtained**.
    - **The objective function w has been minimized with $w_{\min} = 0$ and the artificial variables are in the basis at a zero level (that is, with a zero value)**. This means **an initial BFS has been obtained**.
- We enter Phase-2 if one of the latter cases are obtained.

- **Phase-2**
  - If the second case is obtained at the end of Phase-1, then remove all the rows and columns concerned with the artificial variables and artificial objective function w, from the final tableau of Phase-1. The remaining parts of the table form the initial tableau for the optimization procedure that we shall proceed with. Apply usual simplex method to solve the problem.
  - If the third case is obtained, we delete only those artificial variables which are not in the basis. Then, we apply the usual simplex method. If the artificial variables are at zero-level, in every iteration, then no hindrances shall be there. But what if artificial variables are at a positive-level in some iteration? In such a case, we **DO NOT** use the minimum ratio test to determine which variable leaves the basis. We simply remove the artificial variable from the basis. Then, applying simplex algorithm in the usual manner will help us optimize z, if possible.

Example 1

The following table shows the final table of Phase-1 of some given LPP. The symbols carry their usual meanings. The actual calculations are not made. Just a hypothetical example for a quick explanation. This shall be same for the examples that follow.

	$v \rightarrow$	A_1	A_2	A_3	x_1	x_2	x_3	x_4	x_5	b
$B.V. \downarrow$	$z \rightarrow$	-2.5	1	1	-1.5	0	-1.5	0	0	4.5
	$w \rightarrow$	-1	-1	-1	0	0	0	0	0	0
x_2	n/a	0.5	0	0	1.5	1	0.5	0	0	0.5
x_4	n/a	-0.5	1	0	3.5	0	0.5	0	0	1.5
x_5	n/a	-2.5	0	1	-5.5	0	-1.5	0	1	1.5

(Case 2 of final iteration of Phase-1)

Example 2

	$v \rightarrow$	A_1	A_2	A_3	x_1	x_2	x_3	x_4	x_5	b
$B.V. \downarrow$	$z \rightarrow$	-2.5	0	1	0	0	1.5	0	0	4
	$w \rightarrow$	-1	0	-1	0	0	0	-3	0	0
x_1	n/a	0.5	0	0	1	8	0.5	0	0	3
A_2	n/a	-0.5	1	0	0	5	0.5	0	0	0
x_4	n/a	-2.5	0	1	0	9	-1.5	1	1	5

(Case 3 of final iteration of Phase-1)

Example 3

	$v \rightarrow$	A	x_1	x_2	x_3	x_4	b
$B.V. \downarrow$	$z \rightarrow$	0	5	-3	0	4	5
	$w \rightarrow$	0	-2	-4	0	0	1
A	n/a	1	8	5	0	0	1
x_3	n/a	0	6	4	1	9	3

(Case 1 of final iteration of Phase-1)

First Theorem Minimization of Distance - A Theorem on Existence and Uniqueness

Theorem

Let S be a non-empty closed convex set in $\mathbb{R}^n$ and let $\vec{y} \notin S$. Then there exists a unique point $\vec{x}_0$ in S at a minimum distance from $\vec{y}$.

The Proof of first theorem - Part 1 : Existence

- As S is non-empty, we have $\vec{z} \in S$. As $\vec{y} \notin S$, for every such $\vec{z}$, $\|\vec{y} - \vec{z}\| \neq 0$, giving us $\|\vec{y} - \vec{z}\| > 0$. This represents the distance between $\vec{y}$ and any point $\vec{z}$ in S. We are to show the existence of a point in S which is at a minimum distance from $\vec{y}$.
- Take a fixed $\vec{k}$ in S. Consider a set $S_1 = S \cap \{\vec{x} \in \mathbb{R}^n : \|\vec{y} - \vec{x}\| \leq \|\vec{y} - \vec{k}\|\}$. S is closed and the other set in the definition of S_1 is a closed ball with radius $\|\vec{y} - \vec{k}\|$ and center at $\vec{y}$, which makes it closed. So, S_1 is a closed set, being an intersection of two closed sets. Also, S_1 is a subset of a closed ball, which is bounded. Hence, S_1 is bounded. This shows S_1 is closed and bounded, that is, S_1 is compact.

- Here is the main trick of the proof. We are to find the infimum of $\|\vec{y} - \vec{x}\|$, where $\vec{x}$ is a point in S. Furthermore, we need to show that the infimum is due to a point, say x_0, in S, so that we can say $\|\vec{y} - \vec{x_0}\|$ is the minimum distance between a fixed $y \notin S$, and a point in S. We do it by obtaining such a point in S_1.
- Let us a define a function f on S_1 with codomain $\mathbb{R}$ by $f(\vec{x}) = \|\vec{y} - \vec{x}\|$, for every $\vec{x}$ in S_1. This function is a continuous function. Also, S_1 is a compact set in the metric space $(\mathbb{R}^n, \|\vec{a} - \vec{b}\|)$. So, we must have a point x_0 in S_1 such that $f(\vec{x_0}) = \|\vec{y} - \vec{x_0}\|$ is the minimum value of the function.

- Now let us look back at the definition of S_1. It is a collection of points in S, whose distance is less than or same as $\|\vec{y}-\vec{k}\|$. But, $\|\vec{y}-\vec{k}\|$ is a distance between a point in S and y. If there existed no points other than $\vec{k}$, it would have been minimum. Otherwise, we would've confirmed the existence of other such points. It doesn't matter if we take consideration of how large $\|\vec{y}-\vec{k}\|$ is, we just need this value, or possibly some other value given by some point in S, which is less than this value.
- We have already found that there is a point in S_1, and hence in S, which gives the minimum value of $\|\vec{y}-\vec{x}\|$ $(x \in S)$ whenever it is less than or same as $\|\vec{y}-\vec{k}\|$. If there are any other points $\vec{x}$ in S, with the property that $\|\vec{y}-\vec{x}\| > \|\vec{y}-\vec{k}\|$, the same point $\vec{x_0}$ will work. Hence we have found a point $\vec{x_0}$ in S, such that the distance between $\vec{x_0}$ and $\vec{y}$ is minimum.

The Proof of first theorem - Part 2 : Uniqueness

- To show uniqueness of the point of minimum distance, assume the contrary. So there are two distinct points $\vec{w_1}$ and $\vec{w_2}$ in S such that both of them give the minimum value, which is unique. As S is a convex set, we must have $\dfrac{\vec{w_1}+\vec{w_2}}{2}$ in S.
- We have,

$$\|\frac{\vec{w_1}+\vec{w_2}}{2}-\vec{y}\| = \frac{1}{2}\|\vec{w_1}+\vec{w_2}-2\vec{y}\| \leq \frac{1}{2}(\|\vec{w_1}-\vec{y}\|+\|\vec{w_2}-\vec{y}\|).$$

- Due to uniqueness of minimum value, we have, $\|\vec{w_1} - \vec{y}\| = \|\vec{w_2} - \vec{y}\|$. This gives,
$$\|\frac{\vec{w_1} + \vec{w_2}}{2} - \vec{y}\| \leq \|\vec{w_2} - \vec{y}\|.$$
- If $<$ holds, then it is a contradiction since the definition of minimum is being violated. If we assume that $=$ holds, then we again arrive at a contradiction since, we must have the following existence
$$\exists \lambda \in \mathbb{R} : \vec{y} - \vec{w_2} = \lambda(\vec{y} - \vec{w_1}).$$
On taking norm, we get $\lambda = \pm 1$, which either contradicts the assumption of $\vec{w_1} \neq \vec{w_2}$ or that of $y \notin S$. Hence, the uniqueness is guaranteed.

Second Theorem on Minimization of Distance - A Theorem on Characterization

Theorem

Let S be a non-empty and closed convex set and let $\vec{y} \notin S$. Then, $\vec{x_0}$ is a minimizing point iff $(\vec{y} - \vec{x_0})^t(\vec{x} - \vec{x_0}) \leq 0$, for all $\vec{x} \in S$.

The proof of second theorem - $\Rightarrow$ part

- Let $\vec{x_0}$ be the minimizing point. Then, for all $\vec{x} \in S$, $\|\vec{y} - \vec{x_0}\|^2 \leq \|\vec{y} - \vec{x}\|^2$.
- As $\vec{x}$, $\vec{x_0} \in S$, and as S is convex, $\lambda\vec{x} + (1 - \lambda)\vec{x_0} = \vec{x_0} + \lambda(\vec{x} - \vec{x_0})$ is in S $(0 \leq \lambda \leq 1)$.
- Replacing $\vec{x}$ by this point in the inequality, we get, $\vec{x} \in S$, $\|\vec{y} - \vec{x_0}\|^2 \leq \|\vec{y} - \vec{x_0} - \lambda(\vec{x} - \vec{x_0})\|^2$.
- By definition of a dot product, we have the RHS of this inequality as $\|\vec{y} - \vec{x_0}\|^2 + \lambda^2\|\vec{x} - \vec{x_0}\|^2 - 2\lambda(\vec{y} - \vec{x_0}).(\vec{x} - \vec{x_0})$.

- Combining the last two inequalities, we get, $2\lambda(\vec{y} - \vec{x_0})^t(\vec{x} - \vec{x_0}) \leq \lambda^2\|\vec{x} - \vec{x_0}\|^2$.
- If $\lambda = 0$, then the point obtained convex combination is simply $\vec{x_0}$ and the inequality holds trivially. So let $\lambda > 0$. Dividing the above inequality by λ in this case and taking limits as $\lambda \to 0^+$, we get the required inequality.

The proof of second theorem - $\Leftarrow$ part

Let $(\vec{y} - \vec{x_0})^t(\vec{x} - \vec{x_0}) \leq 0$, for all $\vec{x} \in S$. This means $(\vec{x_0} - \vec{x})^t(\vec{y} - \vec{x_0}) \geq 0$, for all $\vec{x} \in S$. We have, for all $\vec{x}$ in S,

$$\|\vec{y}-\vec{x}\|^2 = \|\vec{y}-\vec{x_0}+\vec{x_0}-\vec{x}\|^2 = \|\vec{y}-\vec{x_0}\|^2+\|\vec{x_0}-\vec{x}\|^2+2(\vec{x_0}-\vec{x})^t(\vec{y}-\vec{x_0})$$

which gives, for all $\vec{x}$ in S,

$$\|\vec{y} - \vec{x}\| \geq \|\vec{y} - \vec{x_0}\|.$$

Hence, $\vec{x_0}$ is the minimizing point.

Any
questions

Thank you